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Renormalisation group treatment of a random Potts model with different numbers of Potts spin states

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Abstract. The mean-field renormalisation group method is used to study a random Potts model which consists of a mixture of p-state and q-state Potts spins. The Hamiltonian is written in terms of a critical and a non-critical variable. The concentration dependence of the critical temperature and non-critical parameter is determined.

The critical behaviour of the q-state Potts model has been the subject of much investigation in recent years (Wu 1982). It is known that for $q \leqslant q_c(d)$ this system undergoes a second-order phase transition with q-dependent values of the critical exponents which describe the statics and dynamics of the approach to criticality.

Various aspects of random Potts models, including the dilute system, have also been studied (Kinzel and Domany 1981). A different generalisation to a random model which consists of two types (p-state and q-state) of Potts spins was considered by Miyazima (1984). This is of particular interest since it enables one to study the crossover between the critical behaviour of the q-state Potts model and the critical behaviour of the p-state Potts model as the concentration of both types of spins is varied.

In this work we apply the mean-field renormalisation group (MFRG) technique (Indekeu *et al* 1982) to go beyond the mean-field-like approximation used by Miyazima.

The Hamiltonian for the random system with a concentration x $(1-x)$ of $p(q)$-component Potts spins can be written, for $p > q$:

$$-\beta\mathcal{H} = K \sum_{\langle ij \rangle} \left(\sum_{k=1}^{p} \eta_i P_i^k \eta_j P_j^k + \sum_{k=1}^{q} [(1-\eta_i)\tilde{P}_i^k (1-\eta_j)\tilde{P}_j^k + (1-\eta_i)\tilde{P}_i^k \eta_j P_j^k + \eta_i P_i^k (1-\eta_j)\tilde{P}_j^k] \right) \tag{1}$$

where η_i takes the value 1 (0) with probability x $(1-x)$ if site i is occupied by a $p(q)$-component Potts spin, and P_i^k ($\tilde{P}_i^k$) is the projection operator onto the kth state of the $p(q)$-component Potts spin. The summation is over the nearest-neighbour pairs and each spin is surrounded by z neighbours.

Let us assume that, below the transition temperature, broken symmetry corresponds to ordering of the first Potts state ($k = 1$), whereas ordering does not distinguish between the $(q-1)$ states of a q-component Potts spin or the remaining $(p-q)$ states of a p-component Potts spin. This suggests the use of parameters s and t in the following way:

$$\begin{aligned} &\langle P^1 \rangle = (1/q)[1+(q-1)s]t \qquad \langle \tilde{P}^1 \rangle = (1/q)[1+(q-1)s] \\ &\langle P^k \rangle = (1/q)(1-s)t \qquad \langle \tilde{P}^k \rangle = (1/q)(1-s) \qquad \text{for } 1 < k \leqslant q \\ &\langle P^k \rangle = (1-t)/(p-q) \qquad \text{for } q < k \leqslant p \end{aligned} \tag{2}$$

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where $\langle\ \rangle$ means a thermodynamic average. $\langle P^k\rangle$ ($\langle\tilde{P}^k\rangle$) represents the fraction of $p(q)$-component Potts spins occupying state k.

In the mean-field approximation each spin is subjected to a field which is the thermodynamic ($\langle\ \rangle$) and configurational ($\bar{}$) average of the field created by its neighbours. Within this approximation the first term of $-\beta\mathcal{H}$ in (1) is written as $(Kz/2)\sum_i \sum_{k=1}^{p} \eta_i P_i^k \eta_j \langle P_j^k\rangle$, and a similar procedure applies to the other terms, resulting after some rearrangement in

$$-\beta\mathcal{H}_{MF} = \tfrac{1}{2}Kz\sum_i \left[\frac{q}{q-1}[\eta_i R_i + (1-\eta_i)\pi_i^1][xR + (1-x)\pi] + \frac{p}{q(p-q)}\left(\eta_i\rho_i + (1-\eta_i)\frac{p-q}{p}\right)\left(x\rho + (1-x)\frac{p-q}{p}\right) + \frac{1}{p}\right] \tag{3}$$

where the concentration dependence arises from the configurational average and

$$R_i = P_i^1 - t_i/q \qquad t_i = \sum_{k=1}^{q} P_i^k \qquad \rho_i = t_i - q/p \qquad \pi_i^1 = \tilde{P}_i^1 - 1/q$$

$$\overline{\eta_i\langle R_i\rangle} = xR \qquad \overline{(1-\eta_i)\langle\pi_i^1\rangle} = (1-x)\pi \qquad \overline{\eta_i\langle\rho_i\rangle} = x\rho$$

$$\pi = [(q-1)/q]s \qquad \rho = t - q/p. \tag{4}$$

Equation (3) can be written in a more condensed form as

$$-\beta\mathcal{H}_{MF} = \tfrac{1}{2}\sum_i \left(\frac{q}{q-1}Q_i h_Q + \frac{p}{q(p-q)}\mathcal{P}_i h_{\mathcal{P}}\right) + \text{constant} \tag{5}$$

with

$$Q_i = \eta_i R_i + (1-\eta_i)\pi_i^1 \qquad \mathcal{P}_i = \eta_i\rho_i + (1-\eta_i)(p-q)/p$$

and where h_Q is the field conjugate to the critical parameter Q and $h_{\mathcal{P}}$ is the field conjugate to the non-critical parameter $\mathcal{P}$.

As a matter of fact, the vanishing of order means in this case that state 1 can no longer be distinguished from the $(q-1)$ other states of a q-component spin; therefore at the critical point one must have $R = \pi = 0$. As for ρ, it measures the deviation of t (the fraction of occupied states in the first group of q states of a p-component spin) from the value q/p that it would take when the group of q states cannot be distinguished from the remaining $(p-q)$ states and they are all occupied with probability $1/p$. This can only happen when all the sites are occupied by p-component Potts spins, i.e. when $x = 1$. Therefore ρ is only critical in this limit, when the model becomes the pure p-state Potts model.

The method of MFRG is based on a comparison of the behaviour of two clusters of different size; the interactions within the clusters are treated exactly and the effect of surrounding spins is simulated by a mean field which is supposed to scale in the same way as the magnetisation of the cluster. We consider here the simple choice of one- and two-site clusters.

The Hamiltonian for the one-site cluster is

$$-\beta\mathcal{H}_1 = \left(\frac{q}{q-1}Q_1 h_Q + \frac{p}{q(p-q)}\mathcal{P}_1 h_{\mathcal{P}}\right) \tag{6}$$

where $h_Q = zKC_Q$, $h_{\mathcal{P}} = zKC_{\mathcal{P}}$.

The mean-field equations are obtained by setting $C_Q = \overline{\langle Q_1 \rangle}$, $C_{\mathscr{P}} = \overline{\langle \mathscr{P}_1 \rangle}$. The Hamiltonian for the two-spin cluster is

$$-\beta' \mathscr{H}_{11} = K' \Bigg(\sum_{k=1}^{p} \eta_1 P_1^k \eta_2 P_2^k + \sum_{k=1}^{q} [(1-\eta_1)\tilde{P}_1^k (1-\eta_2)\tilde{P}_2^k + (1-\eta_1)\tilde{P}_1^k \eta_2 P_2^k + \eta_1 P_1^k (1-\eta_2)\tilde{P}_2^k] \Bigg) + \frac{q}{q-1}(Q_1+Q_2)h'_Q + \frac{p}{q(p-q)}(\mathscr{P}_1+\mathscr{P}_2)h'_{\mathscr{P}} \tag{7}$$

where $h'_Q = K'(z-1)C'_Q$, $h'_{\mathscr{P}} = K'(z-1)C'_{\mathscr{P}}$.

The critical properties of the model are now obtained by assuming that $\overline{\langle Q_1 \rangle}_1$ and $\frac{1}{2}\overline{\langle Q_1 + Q_2 \rangle}_{11}$ scale like the symmetry breaking fields C_Q and C'_Q which, in the vicinity of a second-order phase transition, can be considered very small. We obtain

$$\overline{\langle Q_1 \rangle}_1 = h_Q \left(\frac{x}{q + (p-q)\exp\{K h_{\mathscr{P}}[-1/(p-q) - 1/q]\}} + \frac{1-x}{q} \right) \tag{8a}$$

$$\tfrac{1}{2}\overline{\langle Q_1 + Q_2 \rangle}_{11} = h'_Q \left(x^2 \frac{2\bar{a} + (q-2)\bar{b} + (p-q)\bar{c}}{q\bar{a} + q(q-1)\bar{b} + 2q(p-q)\bar{c} + (p-q)\bar{d} + (p-q)(p-q-1)\bar{e}} + x(1-x)\frac{4\bar{a} + 2(q-2)\bar{b} + (p-q)\bar{c}}{q\bar{a} + q(q-1)\bar{b} + q(p-q)\bar{c}} + (1-x)^2 \frac{2\bar{a} + (q-2)\bar{b}}{q\bar{a} + q(q-1)\bar{b}} \right) \tag{8b}$$

where

$$\bar{a} = \exp[K'(1 + 2h'_{\mathscr{P}}/q)] \qquad \bar{b} = \exp(2K' h'_{\mathscr{P}}/q)$$

$$\bar{c} = \exp[K' h'_{\mathscr{P}}(p-2q)/q(p-q)] \qquad \bar{d} = \exp\{K'[1 - 2h'_{\mathscr{P}}/(p-q)]\}$$

$$\bar{e} = \exp[-2K' h'_{\mathscr{P}}/(p-q)].$$

As for the non-critical parameter we treat it in a self-consistent way ($C_{\mathscr{P}} = \overline{\langle Q_1 \rangle}_1$) and assume $C_{\mathscr{P}} = C'_{\mathscr{P}}$ (de Alcantara Bonfim and Sá Barreto 1985, Plascak and Sá Barreto 1986).

The fixed-point equation obtained from the scaling relation by imposing $K' = K$, $x' = x$ is then

$$z \left(\frac{x}{q + (p-q)\exp\{KzC_{\mathscr{P}}[-1/(p-q) - 1/q]\}} + \frac{1-x}{q} \right) = (z-1) \left(x^2 \frac{2a + (q-2)b + (p-q)c}{qa + q(q-1)b + 2q(p-q)c + (p-q)d + (p-q)(p-q-1)e} + x(1-x)\frac{4a + 2(q-2)b + (p-q)c}{qa + q(q-1)b + q(p-q)c} + (1-x)^2 \frac{2a + (q-2)b}{qa + q(q-1)b} \right) \tag{9}$$

together with

$$C_{\mathscr{P}} = \frac{p-q}{p} \left(1 - x + x \frac{q\exp(KzC_{\mathscr{P}}/q) - q\exp[-KzC_{\mathscr{P}}/(p-q)]}{q\exp(KzC_{\mathscr{P}}/q) + (p-q)\exp[-KzC_{\mathscr{P}}/(p-q)]} \right) \tag{10}$$

where

$$a = \exp\{K + 2K[(z-1)/q]C_{\mathscr{P}}\} \qquad b = \exp\{2K[(z-1)/q]C_{\mathscr{P}}\}$$

$$c = \exp\{K(z-1)[(p-2q)/q(p-q)]C_{\mathscr{P}}\}$$

$$d = \exp\{K - [2/(p-q)](z-1)C_{\mathscr{P}}K\} \qquad e = \exp\{-2K[(z-1)/(p-q)]C_{\mathscr{P}}\}.$$

Equation (9) yields the phase diagram in the temperature/concentration space.

In figure 1 we have plotted $1/K_c(x) \propto T_c(x)$ against concentration for $z = 4$, $p = 3$, $q = 2$. As expected, the transition temperature decreases monotonically from the value predicted by the method for $x = 0$ (pure q-state Potts model, in this case the pure Ising model) to the value predicted for $x = 1$ (pure p-state Potts model). As shown previously (Indekeu *et al* 1982, Marques and Santos 1986), the results of MFRG considerably improve the mean-field estimates.

The case $z = 4$, $p = 6$, $q = 3$ is represented in figure 2. Mean-field approximation predicts a first-order phase transition for the Potts model with $q > 2$, independently of the dimensionality; this agrees with the work of Miyazima (1984) who locates a line of first-order phase transitions in this case. On the other hand, it is known from exact results that in two dimensions the q-state Potts model undergoes a second-order phase transition for $q \leqslant q_c(2) = 4$, so there must be a certain concentration in this random Potts model for which the phase transition changes order. The present MFRG method is only justified in the case of second-order phase transitions. Nevertheless (Indekeu *et al* 1982, Marques and Santos 1986), and as can be seen in figure 2, the estimates it gives for the location of critical couplings in cases $q > q_c(d)$ are still good as compared to the mean-field calculation; we therefore think the phase diagram we have obtained

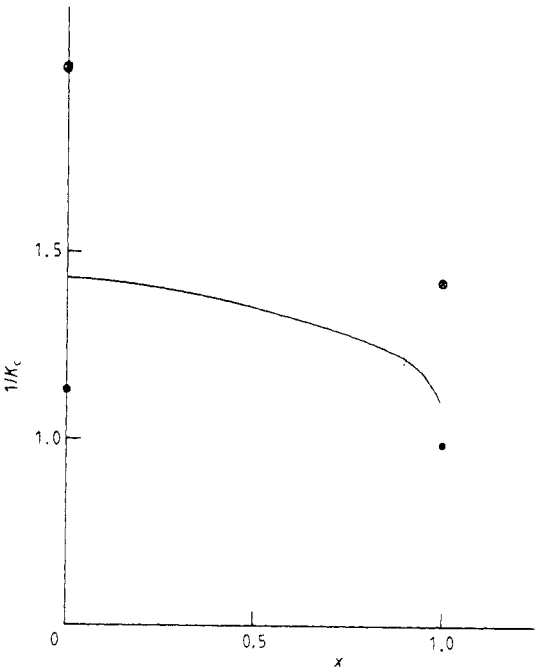

Figure 1. Transition temperature of random Potts model against concentration for $z = 4$, $p = 3$, $q = 2$, in the present approximation. Mean-field predictions for $x = 0$ (pure Ising model) and $x = 1$ (three-state Potts model) are indicated by ⊗, and exact results by ●.

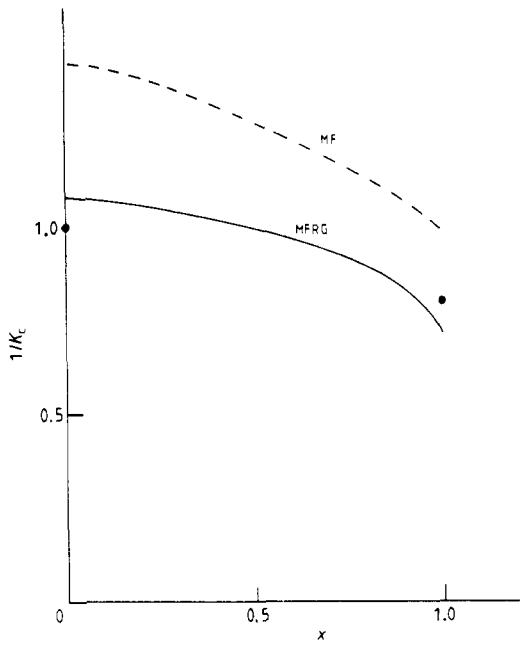

Figure 2. Transition temperature against concentration for $z = 4$, $p = 6$, $q = 3$, as given by the present MFRG calculation and the mean-field (MF) calculation of Miyazima (1984). The full circles represent exact results.

for this model within this approach represents an improvement on Miyazima's mean-field calculation.

In figure 3 we have plotted the variation of ρ as a function of concentration along the transition line. The physically expected behaviour near $x = 1$, when ρ becomes critical, is recovered.

In conclusion, for the sake of the use of the MFRG method, we have reduced the random Potts model with two types of Potts spins to a model with a critical and a non-critical variable. The use of the method has enabled us to obtain a better estimate for $T_c(x)$.

As remarked before (Indekeu *et al* 1982) this method gives critical exponents which are generally less accurate than the critical couplings, even though this can be improved

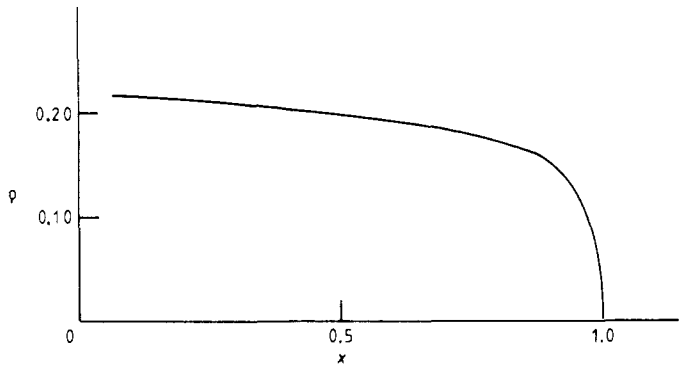

Figure 3. Variation of ρ with concentration along the transition line for $z = 4$, $p = 3$, $q = 2$.

by the use of bigger clusters. In the limit $x = 0$ (corresponding to the Ising model) an estimate for the critical exponent was obtained ($\nu = 1.65$) which agrees with previous work (Marques and Santos 1986). A complete study of the crossover displayed by ν has, however, proved less successful within the present approach, leading to the sort of pathological behaviour already seen in the study of other models (Slotte 1984, Plascak and Sá Barreto 1986). This is probably due to the assumption implicit in (9).

It might well be possible to overcome this difficulty by using a different RG method applied to the Hamiltonian of the model in the form presented here. The study of other critical exponents, namely the dynamical critical exponent z which has been the subject of recent investigations (Lage 1986), would be of great interest. Ways of overcoming the problem concerning the order of the transition are presently being investigated.

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